Estimators

Bias example

Let $Y_i \overset{iid}{\sim} N(\theta, \sigma^2)$

Let $\hat{\theta}_e = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$\begin{aligned}
\text{Bias}(\hat{\theta}_e \mid \theta = \theta_0) &= \mathbb{E}\, \hat{\theta}_e \mid \theta_0 \quad - \theta_0 \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\, Y_i \mid \theta_0 \quad - \theta_0 \\
&= \frac{1}{n} \cdot n\theta_0 \quad - \theta_0 \\
&= 0
\end{aligned}$$

The sample mean is an <u>unbiased</u> estimator of θ.

Variance example

Again $Y_i \overset{iid}{\sim} N(\theta, \sigma^2)$

$$\begin{aligned}
\text{var}(\hat{\theta}_e \mid \theta_0) &= \text{var}(\bar{y} \mid \theta_0) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \text{var}(y_i) \\
&= \boxed{\frac{\sigma^2}{n}}
\end{aligned}$$

MSE example

$$\text{MSE}(\hat{\theta}_e \mid \theta_0) = \underbrace{\frac{\sigma^2}{n}}_{\text{variance}} + \underbrace{0^2}_{\text{bias}^2}$$

Exercise #3 Let $\hat{\theta}_b = w\bar{y} + (1-w)\mu_0$

$$MSE(\hat{\theta}_b | \theta_0) = \mathbb{E}(\hat{\theta} - \theta_0)^2 | \theta_0$$

We could compute variance + bias², but we'll take an alternative approach & compute $\hat{MSE}$ directly.

TRICK: $\mathbb{E}(\hat{\theta}_b - \theta_0)^2 | \theta_0 = \mathbb{E}[\hat{\theta}_b - (\underbrace{w\theta_0 + (1-w)\theta_0}_{\theta_0})]^2 | \theta_0$

$$= \mathbb{E}\left(w(\bar{y}-\theta_0) + (1-w)(\mu_0-\theta_0)\right)^2 | \theta_0$$

$$= \mathbb{E}\, w^2(\bar{y}-\theta_0)^2 + \underbrace{2w(1-w)(\mu_0-\theta_0)\,\mathbb{E}(\bar{y}-\theta_0)}_{0} + (1-w)^2(\mu_0-\theta_0)^2$$

$$= \boxed{w^2 \cdot var(\bar{y}|\theta_0) + (1-w)^2(\mu_0-\theta_0)^2}$$

* Bayesian estimator will have lower MSE than $\hat{\theta}_c$ when prior guess μ_0 is "close to" θ_0.