"Completing the Square"

Let $Y_i \overset{iid}{\sim} \text{normal}(\theta, \sigma^2)$

Let $\theta \mid \sigma^2 \sim \text{normal}(\mu_0, \tau_0^2)$

To prove: $\theta \mid \sigma^2, \vec{y} \sim \text{normal}(\mu_n, \tau_n^2)$

(for some μ_n, τ_n^2) where $\vec{y} = \{y_1, \dots, y_n\}$

Proof:

$$p(\theta \mid \sigma^2, \vec{y}) \propto \underbrace{p(\vec{y} \mid \theta, \sigma^2)}_{\prod_{i=1}^{n} \text{dnorm}(y_i, \theta, \sigma)} \cdot p(\theta \mid \sigma^2)$$

$$\propto \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2\right\} \cdot \exp\left\{\frac{-1}{2\tau_0^2}(\theta-\mu_0)^2\right\}$$

recall $\Sigma y_i = n\bar{y}$

$$\propto \exp\left\{\frac{-1}{2\sigma^2}\left(n\theta^2 - 2n\bar{y}\theta\right) - \frac{1}{2\tau_0^2}\left(\theta^2 - 2\mu_0\theta\right)\right\}$$

$$\propto \exp\left\{\frac{-1}{2}\left[\underbrace{\left(\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}\right)}_{\text{"a"}}\theta^2 - 2\underbrace{\left(\frac{n\bar{y}}{\sigma^2}+\frac{\mu_0}{\tau_0^2}\right)}_{\text{"b"}}\theta\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}a\theta^2 + b\theta\right\}$$

Notice: if $\theta \mid \sigma^2, \vec{y} \sim \text{normal}(\mu_n, \tau_n^2)$ then

$$p(\theta \mid \sigma^2, \vec{y}) = \left(2\pi\tau_n^2\right)^{-1/2} \exp\left\{\frac{-1}{2}\cdot\frac{1}{\tau_n^2}\left(\theta - \mu_n^2\right)\right\}$$

$$\propto \exp\left\{\frac{-1}{2}\cdot\frac{1}{\tau_n^2}\theta^2 + \frac{1}{\tau_n^2}\cdot\mu_n\theta\right\}$$

$$\Rightarrow \boxed{a = \frac{1}{\tau_n}} \quad \& \quad \boxed{\frac{b}{a} = \mu_n}$$